The Max-Distance Network Creation Game on General Host Graphs

D. Bilò, L. Gualà, S. Leucci, G. Proietti

13 Luglio 2012

Introduction

Network Creation Games are games that model the formation of large-scale networks governed by autonomous agents. E.g.:

- Social networks (friendship networks, collaboration networks, scientific citations, ...)
- *Communication networks*, e.g. the Internet

Hosts are rational and egoistic agents who want to buy links in order to construct a good-quality network while spending the least possible.

Not all links can be bought. An host can only buy a link (towards another host) if the link belongs to a given existing network infrastucture.

Problem definition

Given an undirected and connected *host graph* H, and a parameter $\alpha \geq 0$, we define MAXGAME+HG to be a strategic game $\langle V(H), \Sigma, C \rangle$.

The set of strategies of a player $u \in V(H)$ is the power set of all the edges that are incident to u in H

$$\Sigma_u = \mathcal{P}(\ \{(u,v) \in E : v \in V\}\ )$$

Given a strategy profile $\sigma = \langle \sigma_u \rangle_{u \in V}$ we define the graph G_σ as:

$$V(G_\sigma) = V \quad \text{ed} \quad E(G_\sigma) = \bigcup_{u \in V} \sigma_u$$

Intuitively, G_σ contains all the edges bought by the players.

Problem definition

The payoff of a player u w.r.t. σ is a cost composed by the sum of two quantities:

- The *building cost*: α times the number of edges bought by u.
- The *usage cost*: the eccentricity of the vertex u in G_σ.

Each player wants to minimize his cost, so he would like to pursue two conflicting objectives:

- Connect to few other hosts.
- Keep his eccentricy low.

To summarize, the *cost of the player u* is:

$$C_u(\sigma) = \alpha \cdot |\sigma_u| + \varepsilon_\sigma(u)$$

Problem definition

A strategy profile σ is a *Nash Equilibrium* iff. every player cannot decrease his cost by changing his strategy (provided that the strategies of all the other players do not change). More precisely:

Definition (Nash Equilibrium)

A strategy profile σ is a Nash Equilibrium for MAXGAME+HG *if* $\forall u \in V$*:*

$$C_u(\sigma) \leq C_u(\langle \overline{\sigma}_u, \sigma_{-u} \rangle) \quad \forall \overline{\sigma}_u \in \Sigma_u$$

Problem definition

Quite naturally, we can define a *social cost function* as the sum of the player's payoffs:

$$SC(\sigma) = \sum_{u \in V} C_u(\sigma)$$

Let OPT be a strategy profile minimizing $SC(\sigma)$. OPT is said to be a *social optimum*:

$$OPT \in \arg\min_{\sigma} SC(\sigma)$$

Problem definition

Let σ ne a N.E., the quantity $Q(\sigma) = \frac{SC(\sigma)}{SC(OPT)}$ is a measure of its quality.

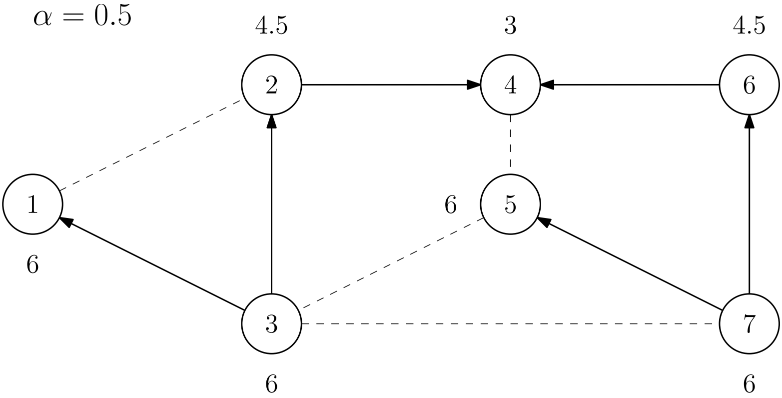
- If $Q(\sigma)$ is 1 then σ is a social optimum.
- If $Q(\sigma)$ is large, then σ has a cost much bigger than a social optimum.

The *Price of Anarchy* (PoA) [?] is the maximum of $Q(\sigma)$ w.r.t. all the NEs:

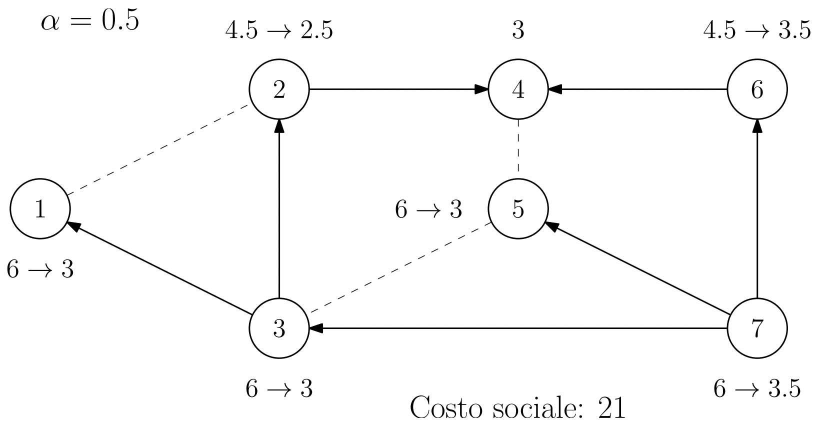
$$PoA = \max_{\sigma : \sigma \text{ is a } N.E.} \frac{SC(\sigma)}{SC(OPT)}$$

Intiutively, the PoA is an upper bound to the quality loss of an equilibrium caused by the egoistic behaviour of the players.

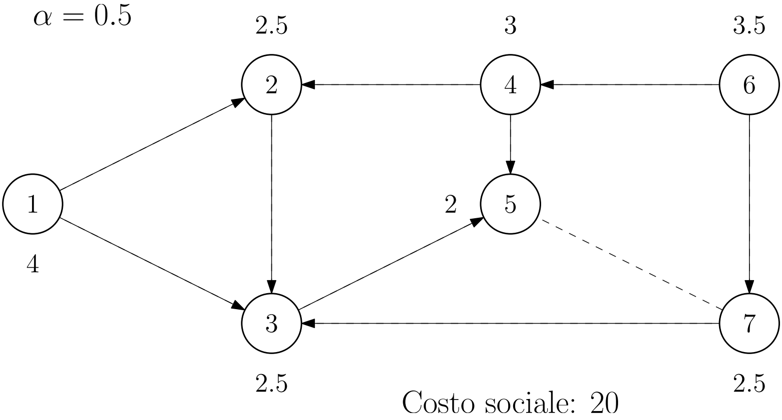
Example: Strategy profile



Example: Equilibrium



Example: Social optimum



Problems of interest for non-cooperative games

- Bound the quality loss of an equilibrium:
  - Price of Anarchy.
- Computational issues.
  - Finding equilibria.
  - Computing the best response of a player w.r.t. the other strategies.
- Dynamics:
  - Analisys of the best and better response dynamics.
  - If the players play in turns, do they end up in a N.E.?
  - Bounding the time needed for convergence.

Related works:

Problems related to MAXGAME+HG can be found in:

- [?] On a network creation game:
  - First model for communication networks.
  - Complete host graph.
  - The usage cost is the routing cost (sum of the distances).
- [?] The price of anarchy in network creation games:
  - Complete host graph.
  - The usage cost is the eccenticity.
- [?] The price of anarchy in cooperative network creation games:
  - Arbitrary host graph.
  - The usage cost is the routing cost (sum of the distances).

MAXGAME+HG was not studied (at the time).

Why Maxgame+HG?

Studying Maxgame+HG is interesting:

- It is a generalization of a prominent problem in the literature.
- Assuming the existence of a complete host graph is unrealistic.
- Insights on how the topology of the host graph affects the quality of the resulting networks.

Results (1/2)

Results and bounds to the Price of'Anarchy:

- Computing the best response of a player is *NP* − *Hard* (reduction from Set-Cover)
- MAXGAME+HG is not a potential game. Moreover the best response dynamics does not converge to an equilibrium if $\alpha > 0$.

Results (2/2)

Results and bounds to the Price of'Anarchy:

Notes	Lower Bound	Upper Bound
–	$\max\{\Omega\left(\sqrt{\frac{n}{1+\alpha}}\right),$ $\Omega(1+\min\{\alpha, \frac{n}{\alpha}\})\}$	$O(\frac{n}{\alpha+r_H})$
$\alpha \geq n$	$\Omega(1)$	$O(1)$
L'equilibrio è un albero	–	$\min\{O(\alpha+1), O(r_H)\}$
$\|E(H)\| = n-1+k$ con $k = O(n)$	–	$O(k+1)$
H è una griglia	$\Omega(1+\min\{\alpha, \frac{n}{\alpha}\})$	$O(\frac{n}{\alpha+r_H})$
H è k-regolare con $k \geq 3$	$\Omega(1+\min\{\alpha, \frac{n}{\alpha}\})$	$O(\frac{n}{\alpha+r_H})$

$n = |V(H)|$, r_H is the raius of H.

Potential game: definition

A game is an *(exact) potential game* if it admits an *(exact) potential* function Φ on $\overline{\Sigma} = \prod_{u \in V} \Sigma_u$ such that, when a player changes his strategy, the change is his payoff is equal to the change of the potential function.

Definition ((Exact) Potential function)

$\Phi : \overline{\Sigma} \to \mathbb{R}$ *is an (exact) potential function if* $\forall \sigma \in \overline{\Sigma}$, $\forall u \in V$, $\forall \overline{\sigma}_u \in \Sigma_u$ *we have:*

$$C_u(\langle \overline{\sigma}_u, \sigma_{-u} \rangle) - C_u(\sigma) = \Phi(\langle \overline{\sigma}_u, \sigma_{-u} \rangle) - \Phi(\sigma)$$

Potential game: properties

- A potential game has at least a N.E.
- The potential function Φ has a global minimum (it is defined on a finite domain).
- Every better response dynamics eventually converges to a N.E. (the value of Φ is monotonically decreasing).

Theorem

For every constant α MAXGAME+HG *is not a potential game. Moreover, if* $\alpha > 0$*, the better response dynamics does not converge to a N.E.*

MAXGAME+HG is not a potential game

Dimostrazione

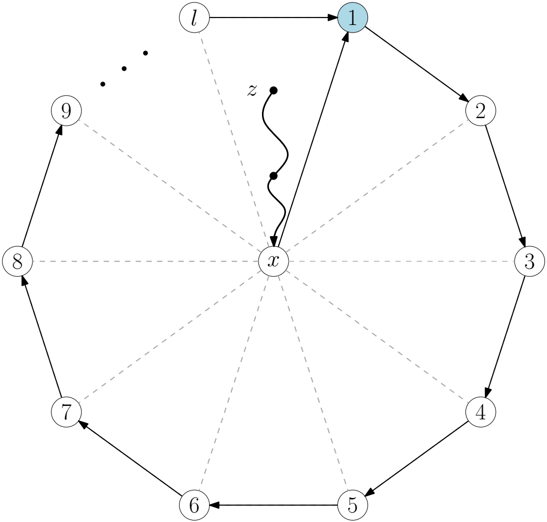
Suppose, by contraddiction, that Φ is a potential function for MAXGAME+HG.

For every value of α we will show an host graph an a sequence of strategy changes that will allow to arbitrarily decrease the value of Φ.

The following *liveness* property will also hold:

There exists a constant $k > 0$ *such that, starting from any turn* t*, every player plays at least once between the turns* $(t+1)$ *and* $(t+k)$.

MAXGAME+HG is not a potential game



Constraints:

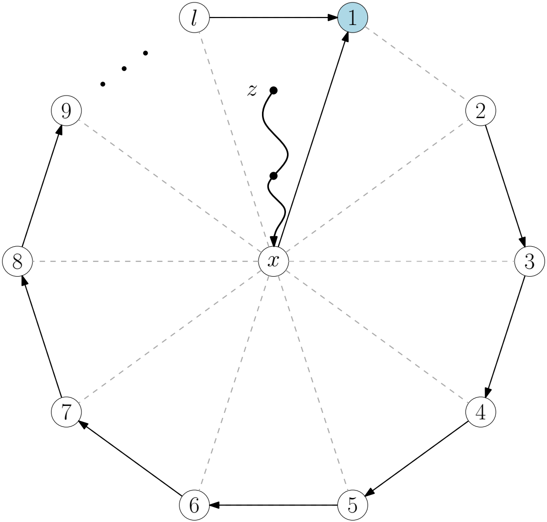
$l > \alpha + 6$

$l \equiv 1 \pmod 3$

$d(x, z) = l - 2$

$C(1) = \alpha + l - 1$

MAXGAME+HG is not a potential game



Constraints:

$l > \alpha + 6$

$l \equiv 1 \pmod 3$

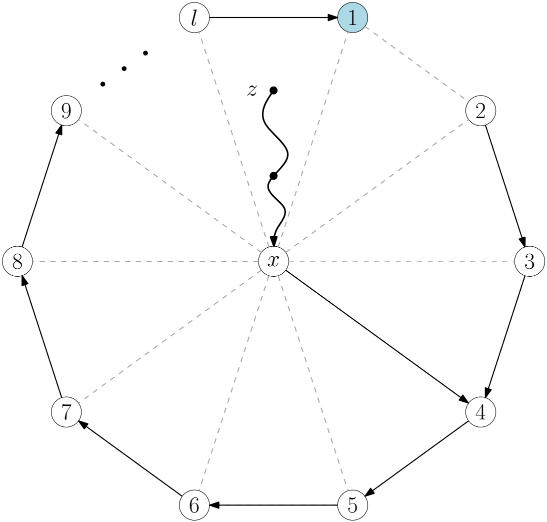
$d(x, z) = l - 2$

$C(1) = l - 1$

Player 1 saves α!

$C(x) = \alpha + l$

MAXGAME+HG is not a potential game



Constraints:

$l > \alpha + 6$

$l \equiv 1 \pmod 3$

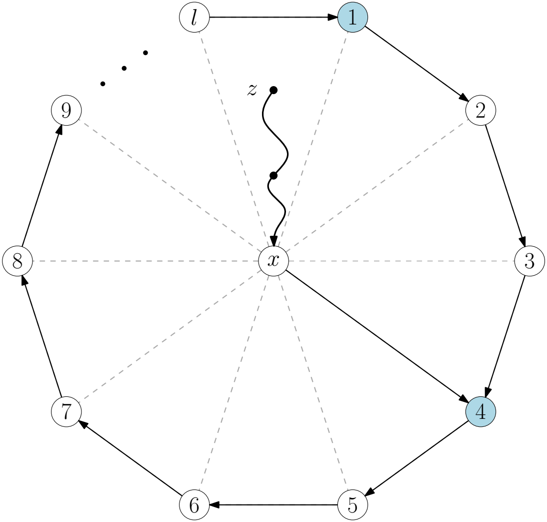
$d(x, z) = l - 2$

$C(x) = \alpha + l - 2$

Player x saves 2!

$C(1) = 2l - 4$

MAXGAME+HG is not a potential game



Constraints:

$l > \alpha + 6$

$l \equiv 1 \pmod 3$

$d(x, z) = l - 2$

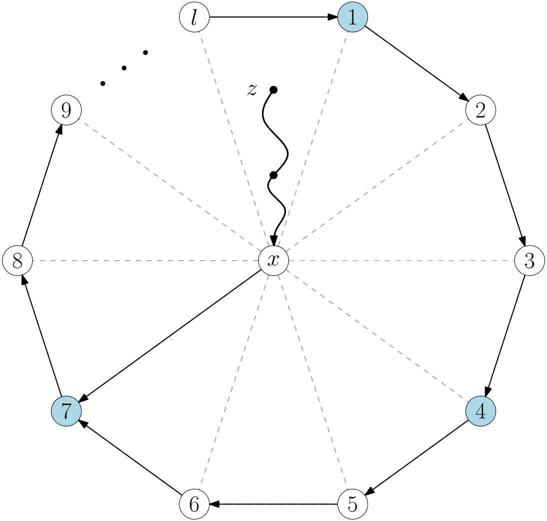
$C(1) = \alpha + l + 2$

Player 1 saves

$l - (\alpha + 6) > 0$

Player 4 is in the same situation of player 1 at the beginning of the game.

MAXGAME+HG is not a potential game



Constraints:

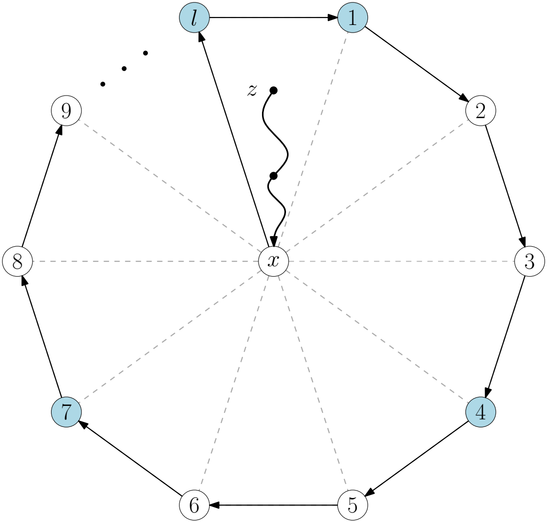
$l > \alpha + 6$

$l \equiv 1 \pmod 3$

$d(x, z) = l - 2$

We can repeat the previous strategy changes. The potential function must decrease by 2 each time.

MAXGAME+HG is not a potential game



Constraints:

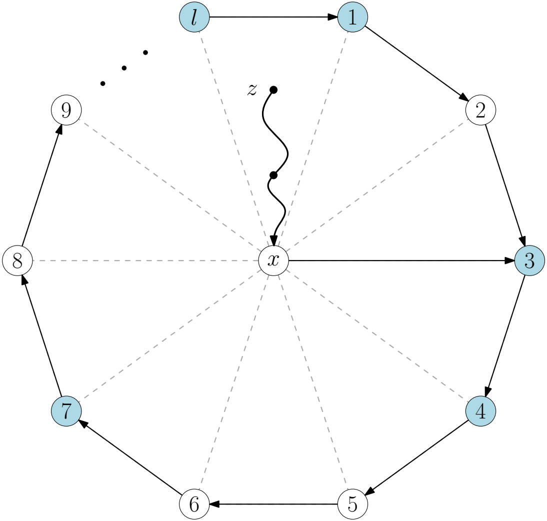
$l > \alpha + 6$

$l \equiv 1 \pmod 3$

$d(x, z) = l - 2$

First, the vertices $1, 4, 7, \ldots l$ will play.

MAXGAME+HG is not a potential game



Constraints:

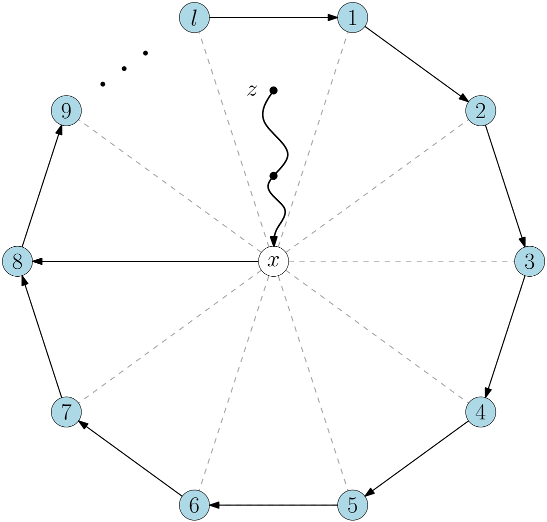
$l > \alpha + 6$

$l \equiv 1 \pmod 3$

$d(x, z) = l - 2$

Then, the vertices $3, 6, 9, \ldots, l - 1$

MAXGAME+HG is not a potential game



Constraints:

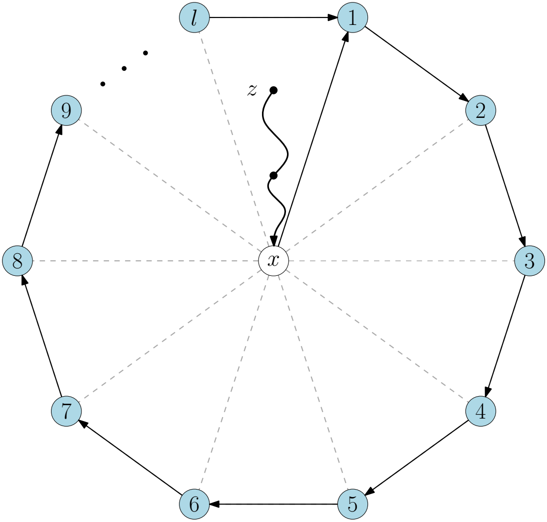
$l > \alpha + 6$

$l \equiv 1 \pmod 3$

$d(x, z) = l - 2$

And finally, the vertices $2, 5, 8, \ldots l - 2$

MAXGAME+HG is not a potential game



Constraints:

$l > \alpha + 6$

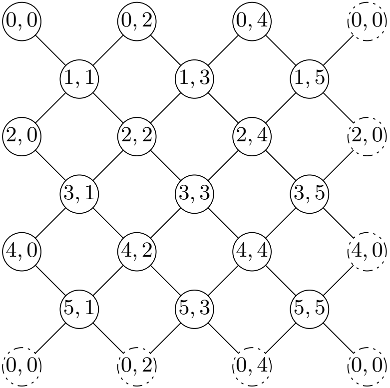
$l \equiv 1 \pmod 3$

$d(x, z) = l - 2$

If we repeat the strategy changes one more time, we return to the starting configuration.

We have a *strategy cycle!*

Lower bounds to the PoA



Definition (Quasi-torus $\overline{H}$)

- *$V(H)$ has $2k^2$ intersection vertices, i.e. the pairs (i,j) s.t. $0 \le i,j < 2k$ and $i+j$ is even.*
- *Each vertex $(i,j) \in V(H)$ has an edge towards $\{(i-1,j-1), (i-1,j+1), (i+1,j-1), (i+1,j+1)\}$ modulo $2k$.*
- *Each edge has weight $\ell = 2(1+\lceil\alpha\rceil)$.*

Some properties

Let $X_{i,j}$ be the set of vertices that are either on the i-th row or on the j-th column. The following properties hold:

i) $\overline{H}$ is vertex-transitive.

ii) The distance between (i, j) and (i', j') is:

$$\ell \cdot \max \left\{ \min \left\{ |i - i'|, 2k - |i - i'| \right\}, \min \left\{ |j - j'|, 2k - |j - j'| \right\} \right\}$$

iii) The eccentricity of every vertex is ℓk.

iv) $\forall 0 \leq i, j \leq 2k$, the distance between a vertex $v \in X_{i,j}$ and $\langle |i - k|, |j - k| \rangle$ is ℓk.

v) If $(u, v) \in E(\overline{H})$ then the eccentricities of u and v in $\overline{H} - \ell$ are at least $\ell(k + 1)$

Building the equilibrium and the Host Graph

Let G be the unweigted graph obtained by replacing each edge (u, v) in $\overline{H}$ by a path of length ℓ between u and v.

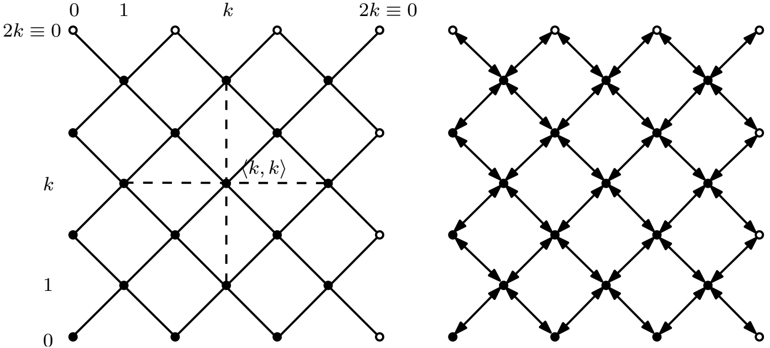
The graph G has $\ell - 1$ new vertices for each edge of H. The total number of vertices is therefore $n = 2k^2 + 4k^2(\ell - 1) = \Theta\left(k^2(1 + \alpha)\right)$ from which $k = \Theta\left(\sqrt{\frac{n}{1+\alpha}}\right)$.

Let σ be a strategy profile such that $G_\sigma = G$ and intersection vertices haven't bought any edges.

Let A be a set containing one vertex per row and one vertex per column. Let H be the graph obtained by adding to G all the vertices in $X_{i,j}$ $\forall\langle i, j\rangle \in A$.

σ is an equilibrium for MAXGAME+HG with host graph H.

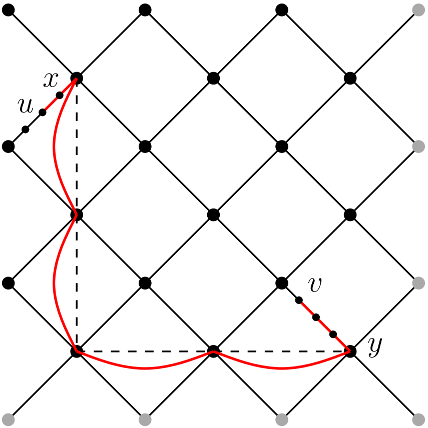
Building the equilibrium and the Host Graph



σ is a Nash Equilibrium

- Every intersection vertex $\langle i, j \rangle$ of G has eccentricity ℓk.
- By property (iv), $\langle i, j \rangle$ cannot improve its eccentricy using the edges towards $X_{i,j}$.
- The vertices $\langle i, j \rangle$ haven't bought edges, so they cannot remove of swap edges.
- The vertices on the paths have an eccentricy of at most $\ell k + \frac{\ell}{2}$ as the nearest intersection vertex is at most at a distance of $\frac{\ell}{2}$.
- The vertices on the paths can only remove a single edge. By property (v), doing so would increase their eccentricity by at least $\ell(k+1) - \ell k - \frac{\ell}{2} > \alpha$.

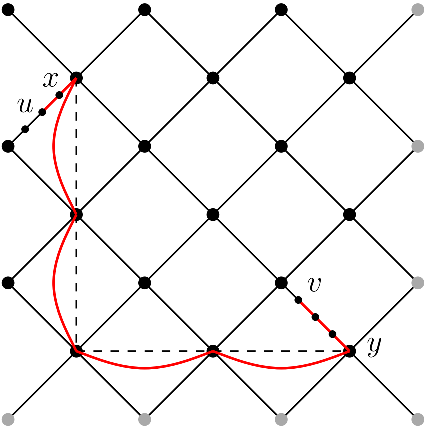
Bounding the PoA



The radius of G is $\Omega(\ell k)$.
The radius of H is $O(\ell)$. A path between two vertices u and v can be built as the union of:

1. A subpath of length $O(\ell)$ between u and an intersection vertex x.
2. A subpath of at most 4 edges in $E(H) \setminus E(G)$ between x and an intersection vertex y "near" v.
3. A subpath of length $O(\ell)$ between y and v.

Bounding the PoA



$$PoA \geq \frac{SC(\sigma)}{SC(OPT)} \geq \frac{SC(G)}{SC(H)}$$

$$= \Omega\left(\frac{\alpha n + n\ell k}{n\ell}\right)$$

$$= \Omega\left(\frac{n\ell k}{n\ell}\right) = \Omega(k)$$

$$= \Omega\left(\sqrt{\frac{n}{1+\alpha}}\right)$$